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EXTENSIONS OF PARANORMAL OPERATORS AND THIER PROPERTIES (Operator Inequalities and Related Area)

AUTHOR(S):

Yanagida, Masahiro; Yamazaki, Takeaki

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EXTENSIONS OF PARANORMAL OPERATORS AND THEIR PROPERTIES

東京理科大 理 柳田 昌宏 (Masahiro Yanagida)
東京理科大 理 山崎 文明 (Takeaki Yamazaki)

ABSTRACT. This report is based on the following papers:

- [Y1] T.Yamazaki and M.Yanagida, *A characterization of log-hyponormal operators via p -paranormality*, Sci. Math. **3** (2000), 19–21.
[Y2] T.Yamazaki and M.Yanagida, *A further generalization of paranormal operators*, Sci. Math. **3** (2000), 23–31.

An operator T is said to be paranormal if $\|T^2x\| \geq \|Tx\|^2$ holds for every unit vector x . Several extensions of paranormal operators have been considered until now, for example, absolute- k -paranormal and p -paranormal operators introduced in [6] and [3], respectively. As a further generalization of paranormal operators, we shall introduce absolute- (p, r) -paranormal operators for $p > 0$ and $r > 0$ such that $\| |T|^p |T^*|^r x \|^r \geq \| |T^*|^r x \|^p$ for every unit vector x . We shall show several properties on absolute- (p, r) -paranormal operators as generalizations of the results on absolute- k -paranormal and p -paranormal operators. We shall also show a characterization of log-hyponormal operators via absolute- (p, r) -paranormality, that is, an invertible operator T satisfies $\log T^*T \geq \log TT^*$ if and only if T is absolute- (p, r) -paranormal for all $p > 0$ and $r > 0$.

1. Introduction

In this report, an operator means a bounded linear operator on a complex Hilbert space H . An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and also T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible.

An operator T is said to be p -hyponormal for $p > 0$ if $(T^*T)^p \geq (TT^*)^p$, and T is said to be log-hyponormal if T is invertible and $\log T^*T \geq \log TT^*$. p -Hyponormal and log-hyponormal operators were defined as extensions of hyponormal ones, i.e., $T^*T \geq TT^*$. It is easily seen that every p -hyponormal operator is q -hyponormal for $p \geq q > 0$ by the celebrated Löwner-Heinz theorem “ $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$,” and every invertible p -hyponormal operator for $p > 0$ is log-hyponormal since $\log t$ is an operator monotone function.

An operator T is said to be paranormal if

$$(1.1) \quad \|T^2x\| \geq \|Tx\|^2 \quad \text{for every unit vector } x.$$

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Paranormal operators have been studied by many researchers, for example, [1][5] and [7]. Particularly, Ando [1] showed that every log-hyponormal operator is paranormal. Afterward, in [6], we gave another simple proof of this result by introducing class A as a new class of operators given by an operator inequality. An operator T belongs to class A if

$$|T^2| \geq |T|^2,$$

where $|T| = (T^*T)^{\frac{1}{2}}$, and we showed that every log-hyponormal operator belongs to class A and every class A operator is paranormal.

In [6], we introduced class $A(k)$ and absolute- k -paranormal operators for $k > 0$ as generalizations of class A and paranormal operators, respectively. An operator T belongs to class $A(k)$ if

$$(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2,$$

and T is said to be absolute- k -paranormal if

$$(1.2) \quad \||T|^kTx\| \geq \|Tx\|^{k+1} \quad \text{for every unit vector } x.$$

It is clear that class $A(1)$ equals class A and absolute-1-paranormality equals paranormality since $\||S|y\| = \|Sy\|$ for any $S \in B(H)$ and $y \in H$. An operator T is said to be normaloid if $\|T\| = r(T)$. We showed inclusion relations among these classes in [6]. Class A and class $A(k)$ operators have been studied in [8][9] and [11].

On the other hand, Fujii, Izumino and Nakamoto [3] introduced p -paranormal operators for $p > 0$ as another generalization of paranormal operators. An operator T is said to be p -paranormal if

$$(1.3) \quad \||T|^pU|T|^px\| \geq \||T|^px\|^2 \quad \text{for every unit vector } x,$$

where the polar decomposition of T is $T = U|T|$. It is clear that 1-paranormality equals paranormality. p -Paranormality is based on the following fact [2]: $T = U|T|$ is p -hyponormal if and only if $S = U|T|^p$ is hyponormal for $p > 0$. Actually, it was shown in [3] that $T = U|T|$ is p -paranormal if and only if $S = U|T|^p$ is paranormal for $p > 0$.

Fujii, Jung, S.H.Lee, M.Y.Lee and Nakamoto [4] introduced class $A(p, r)$ as a further generalization of class $A(k)$. An operator T belongs to class $A(p, r)$ for $p > 0$ and $r > 0$ if

$$(1.4) \quad (|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r},$$

and class $AI(p, r)$ is the class of all invertible operators which belong to class $A(p, r)$. It was pointed out in [11] that class $A(k, 1)$ equals class $A(k)$ for each $k > 0$.

In this report, as a parallel concept to class $A(p, r)$, we shall introduce absolute- (p, r) -paranormality which is a further generalization of both absolute- k -paranormality and p -paranormality. Then we shall generalize the results on absolute- k -paranormal and p -paranormal operators for absolute- (p, r) -paranormal operators. We shall also show a characterization of log-hyponormal operators via absolute- (p, r) -paranormality and p -paranormality.

2. Definition and properties of absolute- (p, r) -paranormal operators

We introduce the following class of operators.

Definition ([Y2]). For positive real numbers $p > 0$ and $r > 0$, an operator T is absolute- (p, r) -paranormal if

$$(2.1) \quad |||T|^p |T^*|^r x||^r \geq |||T^*|^r x||^{p+r} \quad \text{for every unit vector } x,$$

or equivalently,

$$(2.2) \quad |||T|^p |T^*|^r x||^r ||x||^p \geq |||T^*|^r x||^{p+r} \quad \text{for all } x \in H.$$

We remark that the definition of absolute- (p, r) -paranormal operators (2.1) and (2.2) are expressed in terms of only T and T^* , without U which appears in the polar decomposition of $T = U|T|$.

To consider the relations to absolute- k -paranormality and p -paranormality, we show another expression of absolute- (p, r) -paranormality as follows.

Proposition 1 ([Y2]). For each $p > 0$ and $r > 0$, T is absolute- (p, r) -paranormal if and only if

$$(2.3) \quad |||T|^p U|T|^r x||^r \geq |||T|^r x||^{p+r} \quad \text{for every unit vector } x,$$

where the polar decomposition of T is $T = U|T|$.

The following result is easily obtained as a corollary of Proposition 1.

Corollary 2 ([Y2]).

- (i) For each $k > 0$, T is absolute- k -paranormal iff T is absolute- $(k, 1)$ -paranormal.
- (ii) For each $p > 0$, T is p -paranormal iff T is absolute- (p, p) -paranormal.
- (iii) T is paranormal iff T is absolute- $(1, 1)$ -paranormal.

It turns out by Corollary 2 that absolute- (p, r) -paranormality is a further generalization of paranormality than both absolute- k -paranormality and p -paranormality.

Proof of Proposition 1. It is well known that $|T^*|^r = U|T|^r U^*$ for $r > 0$, so that (2.2) is equivalent to the following (2.4):

$$(2.4) \quad |||T|^p U|T|^r U^* x||^r ||x||^p \geq |||U|T|^r U^* x||^{p+r} \quad \text{for all } x \in H.$$

It is also well known that $N(S^r) = N(S)$ for any $S \geq 0$ and $r > 0$. By using this fact, we have $R(|T|^r) \subseteq \overline{R(|T|^r)} = N(|T|^r)^\perp = N(|T|)^\perp = N(U)^\perp$, so that $|||U|T|^r U^* x|| = |||T|^r U^* x||$ for all $x \in H$. Hence (2.4) is equivalent to the following (2.5):

$$(2.5) \quad |||T|^p U|T|^r U^* x||^r ||x||^p \geq |||T|^r U^* x||^{p+r} \quad \text{for all } x \in H.$$

Put $x = Uy$ in (2.5), then we have the following (2.6) since $|T|^r U^* U = |T|^r$:

$$(2.6) \quad |||T|^p U|T|^r y||^r ||Uy||^p \geq |||T|^r y||^{p+r} \quad \text{for all } y \in H.$$

(2.6) yields the following (2.7) since $\|y\| \geq \|Uy\|$ for all $y \in H$:

$$(2.7) \quad \| |T|^p U |T|^r y \|^r \|y\|^p \geq \| |T|^r y \|^{p+r} \quad \text{for all } y \in H.$$

Hence (2.5) implies (2.7). Here we show that (2.7) implies (2.5) conversely. Put $y = U^*x$ in (2.7), then we have

$$(2.8) \quad \| |T|^p U |T|^r U^* x \|^r \|U^* x\|^p \geq \| |T|^r U^* x \|^{p+r} \quad \text{for all } x \in H.$$

(2.8) yields (2.5) since $\|x\| \geq \|U^*x\|$ for all $x \in H$. Hence (2.7) implies (2.5), so that (2.5) is equivalent to (2.7). Consequently, the proof of Proposition 1 is complete since (2.7) is equivalent to (2.3). \square

Proof of Corollary 2. We remark that $\| |S|y \| = \|Sy\|$ holds for any $S \in B(H)$ and $y \in H$.

- (i) Put $p = k > 0$ and $r = 1$ in (2.3), then we have (1.2).
- (ii) Put $r = p > 0$ in (2.3), then we have (1.3).
- (iii) Put $r = p = 1$ in (2.3), then we have (1.1). \square

It was shown in [5] and [7] that if T is invertible and paranormal, then T^{-1} is also paranormal. Here we show the following generalization of this well-known result.

Proposition 3 ([Y2]). *The following assertions hold for each $p > 0$ and $r > 0$:*

- (i) *If T is invertible and absolute- (p, r) -paranormal, then T^{-1} is absolute- (r, p) -paranormal.*
- (ii) *If T is invertible and p -paranormal, then T^{-1} is also p -paranormal.*

Proposition 3 can be considered as a parallel result to the following Proposition A for class $\text{AI}(p, r)$ operators.

Proposition A ([Y2]). *The following assertions hold for each $p > 0$ and $r > 0$:*

- (i) *If T belongs to class $\text{AI}(p, r)$, then T^{-1} belongs to class $\text{AI}(r, p)$.*
- (ii) *If T belongs to class $\text{AI}(p, p)$, then T^{-1} also belongs to class $\text{AI}(p, p)$.*
- (iii) *If T is invertible and belongs to class A, then T^{-1} also belongs to class A.*

We prepare the following lemma to give a proof of Proposition 3.

Lemma 4 ([Y2]). *Let T be an invertible operator. For each $p > 0$ and $r > 0$, T is absolute- (p, r) -paranormal if and only if*

$$(2.9) \quad \| |T|^p x \|^r \| |T|^{-1} x \|^p \geq 1 \quad \text{for every unit vector } x.$$

Proof. (2.2) is equivalent to the following (2.10) by putting $y = |T^*|^{-r} x$ since $R(|T^*|^{-r}) = H$:

$$(2.10) \quad \| |T|^p y \|^r \| |T^*|^{-r} y \|^p \geq \|y\|^{p+r} \quad \text{for all } y \in H.$$

(2.10) is equivalent to the following (2.11):

$$(2.11) \quad \| |T|^p y \|^r \| |T^*|^{-r} y \|^p \geq 1 \quad \text{for every unit vector } y.$$

(2.11) is equivalent to (2.9) since $|T^*|^{-1} = |T|^{-1}$, so that the proof is complete. \square

Proof of Proposition 3.

- (i) Obvious by Lemma 4.
- (ii) Put $r = p > 0$ in (i), then we have (ii) by (ii) of Corollary 2. \square

3. Inclusion relations among the related classes

We cite the following result which plays an important role to give proofs of the results in this section.

Theorem H-M (Hölder-McCarthy inequality [10]). *Let A be a positive operator. Then the following inequalities hold for all $x \in H$:*

- (i) $(A^r x, x) \leq (Ax, x)^r \|x\|^{2(1-r)}$ for $0 < r \leq 1$.
- (i') $\|A^r x\| \leq \|Ax\|^r \|x\|^{1-r}$ for $0 < r \leq 1$.
- (ii) $(A^r x, x) \geq (Ax, x)^r \|x\|^{2(1-r)}$ for $r \geq 1$.
- (ii') $\|A^r x\| \geq \|Ax\|^r \|x\|^{1-r}$ for $r \geq 1$.

Firstly, we show the monotonicity of the classes of absolute- (p, r) -paranormal operators for $p > 0$ and $r > 0$ as generalizations of [4, Theorem 4.1] and [6, Theorem 4].

Theorem 5 ([Y2]). *Let T be absolute- (p_0, r_0) -paranormal for $p_0 > 0$ and $r_0 > 0$. Then T is absolute- (p, r) -paranormal for any $p \geq p_0$ and $r \geq r_0$. Moreover, for each $r \geq r_0$ and unit vector x ,*

$$(3.1) \quad f_r(p) = \| |T|^p |T^*|^r x \|^{\frac{r}{p+r}}$$

is increasing for $p \geq p_0$.

Theorem 5 can be considered as a parallel result to the following Theorem B which states the monotonicity of class $\text{AI}(p, r)$ for $p > 0$ and $r > 0$.

Theorem B ([4]). *If T belongs to class $\text{AI}(p_0, r_0)$ for $p_0 > 0$ and $r_0 > 0$, then T belongs to class $\text{AI}(p, r)$ for any $p \geq p_0$ and $r \geq r_0$.*

Proof of Theorem 5. Assume that T is absolute- (p_0, r_0) -paranormal for $p_0 > 0$ and $r_0 > 0$, i.e.,

$$(3.2) \quad \| |T|^{p_0} |T^*|^{r_0} y \|^{r_0} \geq \| |T^*|^{r_0} y \|^{p_0+r_0} \|y\|^{-p_0} \quad \text{for all } y \in H.$$

Then for each $r \geq r_0$ and unit vector x ,

$$\begin{aligned} & \| |T|^{p_0} |T^*|^r x \|^{r_0} \\ &= \| |T|^{p_0} |T^*|^{r_0} |T^*|^{r-r_0} x \|^{r_0} \\ &\geq \| |T^*|^{r_0} |T^*|^{r-r_0} x \|^{p_0+r_0} \| |T^*|^{r-r_0} x \|^{-p_0} \quad \text{by (3.2)} \\ &= \| |T^*|^r x \|^{p_0+r_0} \| |T^*|^{r-r_0} x \|^{-p_0} \\ &\geq \| |T^*|^r x \|^{p_0+r_0} \| |T^*|^r x \|^{\frac{r-r_0}{r} \cdot (-p_0)} \quad \text{by (i') of Theorem H-M for } \frac{r-r_0}{r} \in [0, 1] \\ &= \| |T^*|^r x \|^{\frac{(p_0+r)r_0}{r}}, \end{aligned}$$

so that we have

$$(3.3) \quad |||T|^{p_0}|T^*|^r x||^{\frac{r}{p_0+r}} \geq |||T^*|^r x||.$$

Hence for each $p \geq p_0$, $r \geq r_0$ and unit vector x ,

$$\begin{aligned} & |||T|^p|T^*|^r x|| \\ & \geq |||T|^{p_0}|T^*|^r x||^{\frac{p}{p_0}} |||T^*|^r x||^{1-\frac{p}{p_0}} && \text{by (ii')} \text{ of Theorem H-M for } \frac{p}{p_0} \geq 1 \\ & \geq |||T|^{p_0}|T^*|^r x||^{\frac{p}{p_0}} |||T|^{p_0}|T^*|^r x||^{\frac{r}{p_0+r} \cdot \frac{p_0-p}{p_0}} && \text{by (3.3)} \\ & = |||T|^{p_0}|T^*|^r x||^{\frac{p+r}{p_0+r}} \\ & \geq |||T^*|^r x||^{\frac{p+r}{r}} && \text{by (3.3),} \end{aligned}$$

so that we have

$$(3.4) \quad |||T|^p|T^*|^r x||^{\frac{r}{p+r}} \geq |||T|^{p_0}|T^*|^r x||^{\frac{r}{p_0+r}} \geq |||T^*|^r x||.$$

(3.4) assures that T is absolute- (p, r) -paranormal for any $p \geq p_0$ and $r \geq r_0$, and for each $r \geq r_0$ and unit vector x , $f_r(p) = |||T|^p|T^*|^r x||^{\frac{r}{p+r}}$ is increasing for $p \geq p_0$. \square

Secondly, we show inclusion relations among the class of absolute- (p, r) -paranormal operators and the related classes.

Theorem 6 ([Y2]). *The following assertions hold for each $p > 0$ and $r > 0$:*

- (i) *Every class $A(p, r)$ operator is absolute- (p, r) -paranormal.*
- (ii) *Every absolute- (p, r) -paranormal operator is normaloid.*

(i) of Theorem 6 is a generalization of [4, Theorem 3.5] and [6, Theorem 4], and (ii) is a generalization of [6, Theorem 5] and the following result.

Theorem C ([4]). *Every p -paranormal operator is normaloid for $p > 0$.*

Proof of Theorem 6.

Proof of (i). Assume that T belongs to class $A(p, r)$ for $p > 0$ and $r > 0$, i.e.,

$$(1.4) \quad (|T^*|^r |T|^{2p} |T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r}.$$

Then for every unit vector x ,

$$\begin{aligned} |||T^*|^r x||^2 &= (|T^*|^{2r} x, x) \\ &\leq ((|T^*|^r |T|^{2p} |T^*|^r)^{\frac{r}{p+r}} x, x) && \text{by (1.4)} \\ &\leq (|T^*|^r |T|^{2p} |T^*|^r x, x)^{\frac{r}{p+r}} && \text{by (i) of Theorem H-M for } \frac{r}{p+r} \in (0, 1) \\ &= |||T|^p|T^*|^r x||^{\frac{2r}{p+r}}, \end{aligned}$$

so that we have

$$(2.1) \quad |||T|^p|T^*|^r x||^r \geq |||T^*|^r x||^{p+r} \quad \text{for every unit vector } x,$$

i.e., T is absolute- (p, r) -paranormal.

Proof of (ii). Assume that T is absolute- (p, r) -paranormal. Put $q = \max\{p, r\} > 0$, then T is absolute- (q, q) -paranormal by Theorem 5, i.e., T is q -paranormal by (ii) of Corollary 2. Hence T is normaloid by Theorem C. \square

4. A characterization of log-hyponormal operators via absolute- (p, r) -paranormality and p -paranormality

In [4], the following result was shown which is a characterization of log-hyponormal operators in terms of class $\text{AI}(p, r)$.

Theorem D ([4]). *The following assertions are mutually equivalent:*

- (i) T is log-hyponormal.
- (ii) T belongs to class $\text{AI}(p, p)$ for all $p > 0$.
- (iii) T belongs to class $\text{AI}(p, r)$ for all $p > 0$ and $r > 0$.

Theorem D states that the class of log-hyponormal operators can be considered as the limit of class $\text{AI}(p, r)$ as $p \rightarrow +0$ and $r \rightarrow +0$ since class $\text{AI}(p, r)$ is monotone increasing for $p > 0$ and $r > 0$ by Theorem B.

Here we shall give a characterization of log-hyponormal operators in terms of absolute- (p, r) -paranormality and p -paranormality.

Theorem 7 ([Y1][Y2]). *The following assertions are mutually equivalent:*

- (i) T is log-hyponormal.
- (ii) T is invertible and p -paranormal for all $p > 0$.
- (iii) T is invertible and absolute- (p, r) -paranormal for all $p > 0$ and $r > 0$.

Theorem 7 states that the class of log-hyponormal operators can be considered as the limit of the class of invertible and absolute- (p, r) -paranormal operators as $p \rightarrow +0$ and $r \rightarrow +0$ since the class of absolute- (p, r) -paranormal operators is monotone increasing for $p > 0$ and $r > 0$ by Theorem 5. It is interesting to remark that class $\text{AI}(p, r)$ and the class of invertible and absolute- (p, r) -paranormal operators can be considered to be parallel to each other, but their limits as $p \rightarrow +0$ and $r \rightarrow +0$ coincide. In fact, Theorem 7 gives a more precise sufficient condition for that an operator T is log-hyponormal than Theorem D by (i) of Theorem 6.

In order to give a proof of Theorem 7, we prepare the following result.

Proposition 8 ([Y2]). *The following assertions hold for each $p > 0$ and $r > 0$:*

- (i) T is absolute- (p, r) -paranormal if and only if

$$(4.1) \quad r|T^*|^r|T|^{2p}|T^*|^r - (p+r)\lambda^p|T^*|^{2r} + p\lambda^{p+r}I \geq 0 \quad \text{for all } \lambda > 0.$$

- (ii) T is p -paranormal if and only if

$$(4.2) \quad |T^*|^p|T|^{2p}|T^*|^p - 2\lambda|T^*|^{2p} + \lambda^2I \geq 0 \quad \text{for all } \lambda > 0.$$

Ando [1] gave a characterization of paranormal operators via an operator inequality as follows: T is paranormal if and only if

$$T^{2*}T^2 - 2\lambda T^*T + \lambda^2 I \geq 0$$

for all $\lambda > 0$. A generalization of this result for absolute- k -paranormal operators was shown in [6, Theorem 6], and Proposition 8 is a further generalization for absolute- (p, r) -paranormal operators.

We use the following well-known fact in the proof of Proposition 8.

Lemma E. For positive real numbers $a > 0$ and $b > 0$,

$$\lambda a + \mu b \geq a^\lambda b^\mu$$

holds for $\lambda > 0$ and $\mu > 0$ such that $\lambda + \mu = 1$.

Proof of Proposition 8.

Proof of (i). (2.2) is equivalent to the following (4.3):

$$(4.3) \quad (|T^*|^r |T|^{2p} |T^*|^r x, x)^{\frac{r}{p+r}} (x, x)^{\frac{p}{p+r}} \geq (|T^*|^{2r} x, x) \quad \text{for all } x \in H.$$

By Lemma E,

$$\begin{aligned} (|T^*|^r |T|^{2p} |T^*|^r x, x)^{\frac{r}{p+r}} (x, x)^{\frac{p}{p+r}} &= \left\{ \lambda^{-p} (|T^*|^r |T|^{2p} |T^*|^r x, x) \right\}^{\frac{r}{p+r}} \left\{ \lambda^r (x, x) \right\}^{\frac{p}{p+r}} \\ &\leq \frac{r}{p+r} \cdot \lambda^{-p} (|T^*|^r |T|^{2p} |T^*|^r x, x) + \frac{p}{p+r} \cdot \lambda^r (x, x) \end{aligned}$$

holds for all $x \in H$ and $\lambda > 0$, so that (4.3) implies the following (4.4):

$$(4.4) \quad \frac{r}{p+r} \cdot \lambda^{-p} (|T^*|^r |T|^{2p} |T^*|^r x, x) + \frac{p}{p+r} \cdot \lambda^r (x, x) \geq (|T^*|^{2r} x, x)$$

for all $x \in H$ and $\lambda > 0$.

Conversely, (4.3) follows from (4.4) by putting $\lambda = \left\{ \frac{(|T^*|^r |T|^{2p} |T^*|^r x, x)}{(x, x)} \right\}^{\frac{1}{p+r}} > 0$ in case

$(|T^*|^r |T|^{2p} |T^*|^r x, x) \neq 0$, and letting $\lambda \rightarrow +0$ in case $(|T^*|^r |T|^{2p} |T^*|^r x, x) = 0$. Hence (4.3) is equivalent to (4.4). Consequently, the proof of Proposition 8 is complete since (4.4) is equivalent to (4.1).

Proof of (ii). Put $r = p > 0$ and replace λ^p with λ in (i), then we have (ii) by (ii) of Corollary 2. \square

Proof of Theorem 7. It is pointed out in [4] that every log-hyponormal operator belongs to class $\text{AI}(p, r)$ for all $p > 0$ and $r > 0$, so that (i) \implies (iii) holds by (i) of Theorem 6. And (iii) \implies (ii) can be proved by putting $r = p > 0$ by (ii) of Corollary 2. Hence we have only to prove (ii) \implies (i).

Assume that T is p -paranormal for all $p > 0$. By (ii) of Proposition 8, (4.2) holds particularly for $\lambda = 1$, that is,

$$(4.5) \quad |T^*|^p |T|^{2p} |T^*|^p - 2|T^*|^{2p} + I \geq 0 \quad \text{for all } p > 0.$$

Since T is invertible, (4.5) can be rewritten as the following (4.6):

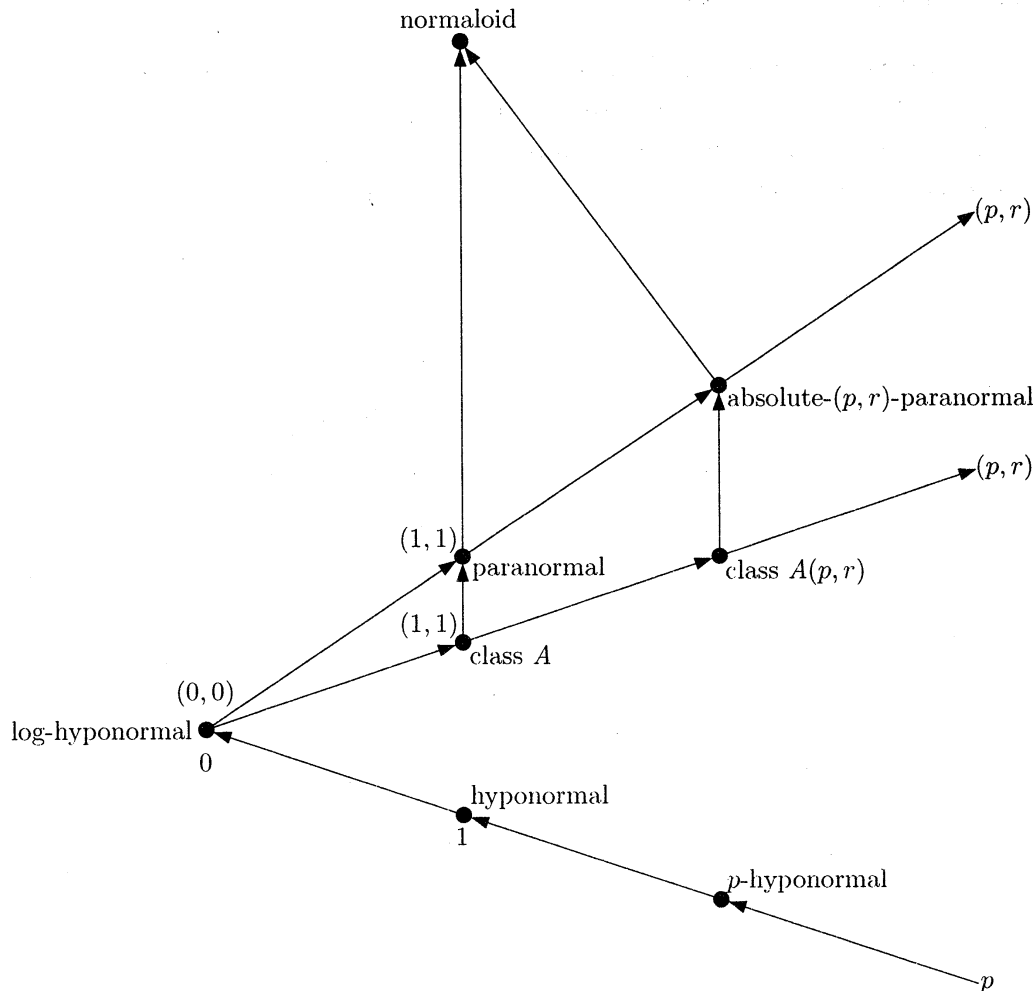
$$(4.6) \quad \frac{|T|^{2p} - I}{p} \geq \frac{|T^*|^{-2p} - I}{-p} \quad \text{for all } p > 0.$$

By letting $p \rightarrow +0$ in (4.6), we have

$$\log |T|^2 \geq \log |T^*|^2,$$

i.e., T is log-hyponormal. □

The following diagram represents the inclusion relations among the classes discussed in this report.



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DEPARTMENT OF APPLIED MATHEMATICS, FACULTY OF SCIENCE, SCIENCE UNIVERSITY OF TOKYO,
1-3 KAGURAZAKA, SHINJUKU, TOKYO 162-8601, JAPAN

E-mail address, Masahiro Yanagida: yanagida@am.kagu.sut.ac.jp

E-mail address, Takeaki Yamazaki: tyamaz@am.kagu.sut.ac.jp